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Indeterminacy in Generative Models: Characterization and Strong Identifiability

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Problem Setting

Latent variables suffer some degree of **indeterminacy** in most generative models.

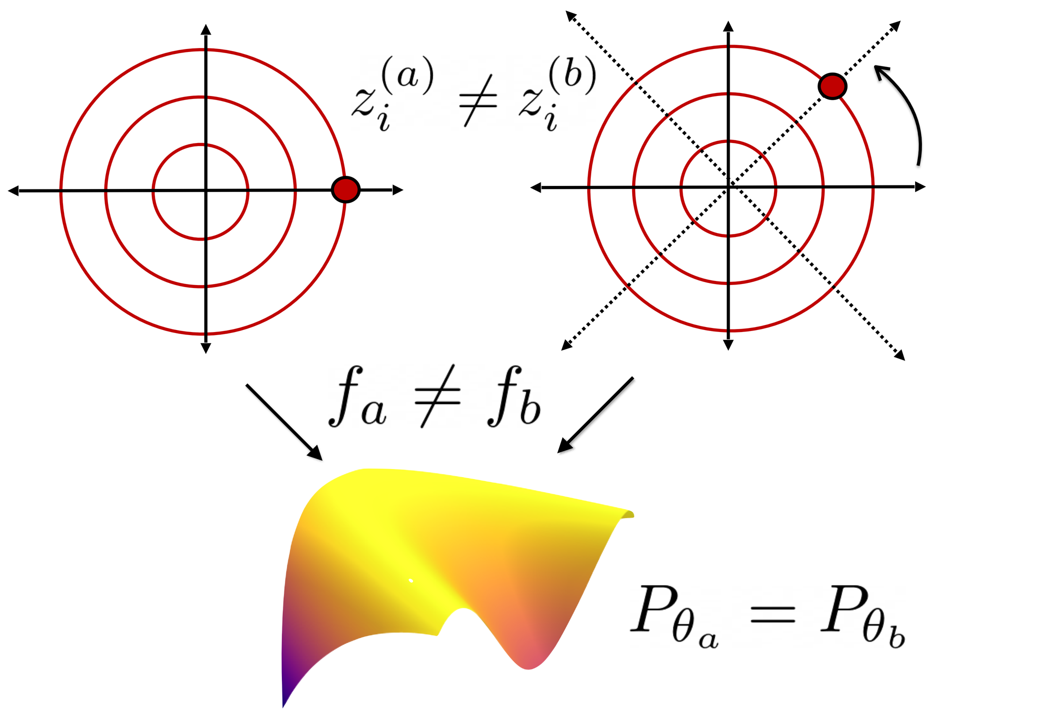
Mathematical setting:

$z_i \sim P_z$ $\qquad$ $\theta = (f, P_z)$ ← Injective

$x_i = f(z_i) + \epsilon_i$ $\qquad$ $\Theta = \mathcal{F} \times \mathcal{P}_z$

↑ observations w/ implied dist. P_θ

↑ model design, i.e., parameter space



Prototypical example: rotational indeterminacy yield equivalent Gaussian generative models.

Questions…

Previous works reduce indeterminacy equivalence classes to some degree:

Model Designs	Indeterminacies
Non-linear, Exponential family	Affine + pointwise non-linearity [Khe2020]
Non-linear, conformal	Permutation + scaling + offset [Hyv1999, Buc2022].
Linear, non-Gaussian	Permutation + scaling [Com1994].

e.g., scaling and permutation indeterminacies are "as good as it gets" in ICA designs.

Strategies to achieve this include:

- Specific model design/parameter space
- Auxiliary information, multi-view

Q1. Do these strategies generalize?

Q2. Is strong identifiability attainable?

$$P_{\theta_a} = P_{\theta_b} \implies \theta_a = \theta_b$$

Q3. If not, what is **"good enough"** for practical uses?

Answers/Contributions

- **Sources of Indeterminacy:** We provide an abstract theoretical framework characterizing the sources of indeterminacies in generic generative models, unifying existing strategies (Q1).
- **Applying the Framework:** We characterize indeterminacies in some specific instances, and propose sufficient conditions for strong identifiability (Q2).
- **Task Identifiability:** We further motivate the study of identifiable generative models via the identifiability of downstream tasks, and provide sufficient conditions for identifiable task outputs (Q3).

Sources of Indeterminacy

It is well-known that measure-preserving transformations result in indeterminacy.

The key message in our main theorem is that they are the only possibility.

Two distinct sources of indeterminacy in parameter space: $\Theta = \mathcal{F} \times \mathcal{P}_z$

- Contribution from generator class.
- Contribution from latent distribution class.

e.g., $\Theta = \{\text{Linear}\} \times \{\text{non-Gaussian}\}$

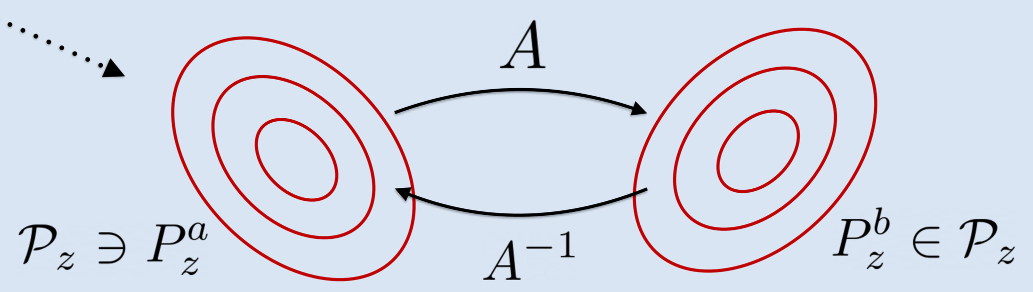
Indeterminacy must be linear, and preserve non-Gaussians: permutation + scalings!

Theorem 2.2 (Cartoon version):

All indeterminacies can be characterized as certain latent transformations $A = f_b^{-1} \circ f_a$:

$$\begin{array}{ccc} z & \xrightarrow{\quad f_a \quad} & x \\ A(z) & \xleftarrow{\quad f_b^{-1} \quad} & x \end{array}$$

(1) Indeterminacies must be formed by pushing and pulling along possible generators.



(2) Indeterminacies must be measure transports between possible latent distributions.

This phenomenon generalizes to all generative models in the specified form!

Applying the Framework

Our framework explains the success of:

Multiple Views: Heterogenous views generated by $\mathcal{F}^e$ with shared latents:

A must satisfy ***(1)*** for each $\mathcal{F}^e$!

Auxiliary Information: Heterogenous environments $\mathcal{P}_{z,e}$ with shared generator:

A must satisfy ***(2)*** for each $\mathcal{P}_{z,e}$!

Fixed Latent Distributions

A novel insight via our framework: fixing latent distributions enable strongly identifiable generative models.

$$\mathcal{P}_z = \{P_z\} \implies A_\# P_z = P_z$$

↑ Measure-preserving automorphisms

Can combine with auxiliary information: iVAE [Khem20] with ExpFam priors fixed a-priori are strongly identifiable**.**

Triangular Monotone Mappings

Triangular monotone (TM) maps are mappings between $\mathbf{Z} = \mathbf{X} = \mathbb{R}^d$.

$$f(x) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \end{bmatrix} \begin{array}{l} \text{Monotone in } x_1 \\ \text{Monotone in } x_2 \\ \end{array}$$

Uniquely specifies CDF transformations. Implications within our framework:

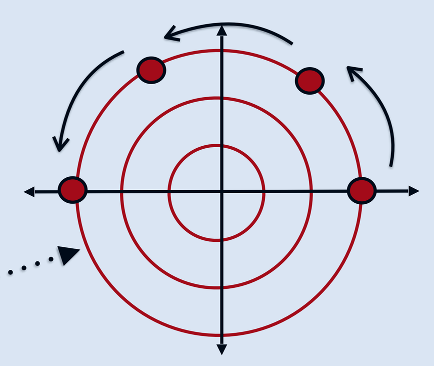
- With fixed distributions, TM generators are **strongly identifiable**.
- TM generators with independent latents are identifiable in the ICA sense.

Task Identifiability

Are weakly identifiable models still useful?

It depends on our task t.

$t(\theta, x, z) = t(A\theta, x, A(z))$



Task outputs constant over indeterminacy "orbits"

Strong ID: OK for any task.

✘ *Tasks using sparsity*
✔ *Tasks using norms*

Causal Discovery using nonlinear ICA

Independence testing of observed and latent variables inferred via ICA [Mon2020].

Typical ICA ID: arbitrary component-wise reparametrizations, no additional mixing.

Independence preserving → task is identifiable, results are consistent.

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